PHILOSOPHICAL TRANSACTIONS.

I. *Researches on the Tides.—Fourth Series.** *On the Empirical Laws of the Tides in the Port of Liverpool.* *By the Rev.* W. Whewell, *M.A. F.R.S.*

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1. In the Philosophical Transactions for 1831 Mr. Lubbock published the results of a discussion of nineteen years of the London tide observations; and from the materials there given, I endeavoured to obtain the mathematical laws of the inequalities of the phenomena, in a memoir which was published in the Transactions for 1834. Mr. Lubbock having now, in Part II. of the Transactions for the present year, published the results of a similar discussion of nineteen years of the Liverpool tide observations, I intend in the present paper to use these results in testing and improving the formulæ to which I was led by the London observations.

Perhaps the precise object of such investigations as this may be best understood by comparing them with corresponding steps in the history of other parts of astronomy; as, for instance, in the progress of our knowledge respecting the Moon's motions. After Hipparchus had singled out and reduced to rule the great inequality of the Moon's motion, the *Equation of the Centre*, it was the employment of succeeding astronomers, as, for instance, Ptolemy and Tycho, to discover, by examination of long-continued observations, other smaller inequalities, and the laws which they follow; as the *Variation*, *Evection*, and others. In the same manner, the great inequality of the tides, the *Semimenstrual Inequality* of the time, is now well understood; and the agreement which Mr. Lubbock showed to exist between the London observations and the formulæ leaves nothing to desire. But formulæ for the observed effects of *lunar* and *solar parallax* and *declination* (although some such formulæ may have been

* For convenience of reference I shall take the liberty of thus numbering the papers in the Philosophical Transactions in which I have attempted to make out the Laws of the Tides. The preceding papers are,

First Series. Essay towards a First Approximation to a Map of Cotidal Lines.—1833, Part I.

Second Series. On the Empirical Laws of the Tides in the Port of London.—1834, Part I.

Third Series. On the Results of Tide Observations made in June 1834 at the Coast Guard Stations in Great Britain and Ireland.—1835, Part I.

employed by the calculators of Tide Tables) have never been published, as far as I am aware, except in the memoir already mentioned. It was therefore a matter of great interest to examine whether the formulæ obtained from the London tides are confirmed by those of Liverpool, and whether any further light is thrown upon the subject by this addition to our materials.

2. The results of this examination have been very satisfactory. The Liverpool observations have both confirmed, in general, my formulæ, and have given me the means of very much improving them. The corrections for lunar parallax and declination, which, as far as they depended on the former investigation, might be considered as in some measure doubtful, and probably only locally applicable, have been so fully verified as to their general form, that I do not conceive any doubt now remains on that subject; and the nature of the local differences in the constants of the formula has also in part come into view. This investigation shows, that notwithstanding the great irregularities to which the tides are subject, the results of the *means* of large masses of good observations agree with the formulæ with a precision not far below that of other astronomical phenomena; as, for example, a fraction of a minute in the times, and a fraction of an inch in the heights.

This precision is the more worthy notice, because the formulæ which we obtain point directly to a very simple general law of the tides; namely, that the tide at any place occurs in the same way as if the ocean imitated the form of equilibrium corresponding to a certain antecedent time. This Equilibrium-Theory (the constant quantities which it introduces being suitably modified,) expresses, with very remarkable exactness, most of the circumstances in my results: I will therefore, before stating them, explain it a little further.

3. The theoretical formula for the position of the pole of the equilibrium-spheroid is

$$\tan 2\,\theta' = -\frac{h \sin 2\,\phi}{h' + h \cos 2\,\phi},$$

where h and h' are the elevation of the spheroid due to the sun and the moon respectively, ϕ the angular distance of the moon from the sun, θ' the angular distance of the pole of the spheroid from the moon's place.

In the case of the tides, we may suppose the actual ocean-spheroid to follow the equilibrium-spheroid at an angular distance λ', the spheroid being that which corresponds to a distance of the sun and moon $\phi - \alpha$, instead of ϕ. Thus we have

$$\tan 2\,(\theta' - \lambda') = -\frac{h \sin 2\,(\phi - \alpha)}{h' + h \cos 2\,(\phi - \alpha)}.$$

In the same manner the theoretical height of the pole of the equilibrium-spheroid above the mean surface is $\sqrt{\{h'^2 + h^2 + 2\,h\,h' \cos 2\,\phi\}}$; and on the equilibrium-theory the height of the tide above the mean surface is $\sqrt{\{h'^2 + h^2 + 2\,h\,h' \cos 2\,(\phi - \alpha)\}}$.

By assuming properly the values of h and h', α and λ', these expressions may be made to agree very closely with the mean results of observation. This was shown with respect to the expression for the time by Mr. Lubbock from the London observations. The Liverpool observations give a still closer agreement, assuming $\lambda' = 11^h\,6^m$, $\alpha = 1^h\,15^m$, $\frac{h}{h'} = \sin 89^m = \sin 22°\,15'$.

The expression for the heights also agrees very nearly with observation, as I shall show, but for this purpose we must suppose $\alpha = 1^h$, $\frac{h}{h'} = \sin 23°\,30'$.

The agreement in these cases is the more remarkable, on account of the want of symmetry in the functions which thus occur. The curve, the ordinate of which represents the time of high water (reckoned from the moon's transit), is not symmetrical with regard to its maximum ordinates. The curve, the ordinate of which represents the height of high water, is not symmetrical with regard to its mean line of abscissas. Yet in both these cases the theoretical and observed curve agree within a minute and an inch during their whole course. It is impossible to doubt, under these circumstances, that the theoretical formula truly represents the observed facts.

4. But this agreement belongs to the mean of all the observations; and we have further to seek for the alteration in the formula, which is requisite in order to represent the effect of changes in the parallax and declination of the sun and moon. In these respects also we find a near agreement of the theory and observation. By the equilibrium-theory, the height h' of the lunar tide ought to be proportional to the cube of the moon's parallax; it is exactly or nearly so: by the same theory the height h' ought to diminish when the moon's declination increases, and by a quantity proportional to the square of the sine of the declination. It is found to do so with great precision.

5. But the equilibrium-theory, since it does not point out the existence of the quantities λ' and α, does not indicate what changes these quantities may be expected to undergo, when the moon's force is altered by the effects of parallax and declination. We find that in that case, these quantities also are altered, and the resulting change in the phenomena may be conceived in the following manner.

If we suppose the moon to revolve about the earth by the diurnal motion, perpetually drawing the waters towards the position of equilibrium, we may conceive that the ocean would form a spheroid, the pole of which would revolve round the earth, following the moon at a certain distance of terrestrial longitude. For the sake of distinctness, let this distance be called the *Retroposition* of the theoretical tide *in longitude*. Its mean value is what I have termed in other communications the "*corrected establishment*" of a place in the open ocean.

If, from an original equilibrium-tide, a derivative tide were sent off, along any channel in which it is no further influenced by the forces of the moon and sun, it would take a certain time in reaching any place in that channel; and the circumstances of the tide at that place would not depend upon the positions and distances of the moon and sun at the time when the tide happens, but upon the positions and distances of those luminaries at a certain time, anterior to the time of the tide by the interval occupied in the transmission of the tide along the channel. Let this interval of time be called the *Retroposition* of the theoretical tide *in time*. It is what, on former occasions, I have called the "*age of the tide*."

6. This phraseology being adopted, the phenomena of the Liverpool tides may be expressed as follows.

(1.) *The effects which the changes of the Moon's force produce upon the Tides, are the same as the effects which those changes would produce upon a Retroposited Equilibrium-tide.*

(2.) *The Retroposition of the Tide in longitude is affected by small changes, which changes are proportional to the variations in the moon's tidal force.*

(3.) *The Retroposition of the Tide in time is also affected by small changes, which changes depend on the variations of the moon's force.*

7. The former of these propositions is proved by the verification of the formulæ already mentioned, since these agree with the formulæ for equilibrium-tides, except in the circumstance of having $\phi - \alpha$ for ϕ. Now this difference is equivalent to a retroposition of the tide in time, of such magnitude that, during this time, the distance of the sun and moon is changed from $\phi - \alpha$ to ϕ. If α be $1^h\ 15^m$, as collected from the law of the times, the retroposition in time is the time requisite for the moon to increase its right ascension from the sun by $1^h\ 15^m$; that is, it is $\frac{75}{48}$ days nearly, or 1 day $13\frac{1}{2}$ hours. The tide at Liverpool agrees nearly with an equilibrium-tide produced in the southern ocean, $37\frac{1}{2}$ hours previously to the moon's transit at that port, and transmitted thither unchanged.

8. The second of the above propositions is proved by tracing the effect of changes of lunar parallax and declination upon the results compared with the above formula for the times and heights, namely,

$$\tan 2(\theta' - \lambda') = -\frac{h \sin 2(\phi - \alpha)}{h' + h \cos 2(\phi - \alpha)} \quad \ldots \ldots \quad (a.)$$

$$y = \sqrt{\{h'^2 + h^2 + 2\,h\,h' \cos 2(\phi - \alpha)\}} \quad \ldots \ldots \quad (b.)$$

By the equilibrium-theory, the change which would be produced by any alteration of the moon's force would correspond to the effect of an alteration in the value of h', the amount of the lunar tide. It appears from the examination of the observations, that this change takes place in fact, but that we must also suppose a change in λ' in order that the formula (*a.*) may represent the observed intervals of time. This change in λ', the retroposition of the tide in longitude, is

$2^m{\cdot}5\ (p - 57)$ for parallax p minutes. (See Art. 15.)
$84^m \sin^2 \delta$ for declination δ. (Art. 21.)

Now, by the theory, the effect of a change in the moon's parallax on the equilibrium-tide is as the change of parallax; and the effect of the moon's declination is a change proportional to the square of the declination. Therefore the second of the above three propositions is established.

According as the moon's parallax is less, and according as her declination is greater, the moon's tidal force is less, and h' in the above formulæ is less. Yet it is remarkable that these two circumstances affect the magnitude of the retroposition of the tide in opposite ways. In one case λ' is augmented, in the other case it is diminished. When the moon's force decreases, by her receding further from the earth, the tide follows the moon at a greater interval; the mean interval increasing from $10^{h}\ 55^{m}$ to $11^{h}\ 12^{m}$, while the parallax diminishes from $61'$ to $54'$. But when the moon moves away from the equator, which also diminishes her tidal force, the tide follows her more closely, the interval decreasing from $11^{h}\ 12^{m}$ to $10^{h}\ 55^{m}$, while the declination increases from $0°$ to $27°$.

The Liverpool tide happens about 11 hours after the next preceding transit; and as the retroposited tide happens about $37\frac{1}{2}$ hours before this transit, we must suppose the Liverpool tide to be produced at an interval of $48\frac{1}{2}$ hours preceding the time at which it is observed, in order to make it agree nearly with the equilibrium-theory; and we may suppose this time to be employed in the transmission of the tide along its channel. If we suppose the original tide to lag behind the position of equilibrium, we may suppose the amount by which it lags to vary with the changes of the moon's force, to the amount above stated as the variation of λ'. On this supposition we may suppose the time of transmission of the tide along its channel to be constant. Or we may suppose that the changes of the moon's force not only affect the lagging of the original tide behind the equilibrium position, but also affect the velocity of transmission to Liverpool. In either of these ways the circumstances of the tide may be hypothetically represented; but it will, of course, be understood that we use such hypotheses at present only for the sake of connecting and representing the facts.

9. The effect which changes in the moon's force produce upon the retroposition of the tide in time, that is, on the value of α in the formulæ (*a.*) and (*b.*), is more difficult to determine with any precision. It is, however, manifest from the general course of the quantities in the Tables, that α is greater as the moon's parallax is greater, and as her declination is greater. This is proved by each Table independently. Thus I have collected as the amount of this change,

$2^{m}{\cdot}5\ (p-57')$ from the effect of parallax on the times (Art. 18.),
$4^{m}\ (p-57')$ from the effect of parallax on the heights (Art. 20.),
$75^{m}\sin^{2}\delta$ from the effect of declination on the times (Art. 23.);

the effect of declination on the heights offers no clear evidence of a change in α.

Since, in the change of parallax from $54'$ to $61'$, the value of α, as given by the times, changes from about $1^{h}\ 8^{m}$ to $1^{h}\ 24^{m}$, the retroposition of the tide in time varies from about 34 to 42 hours.

10. The circumstances of the Liverpool tides may be represented hypothetically

in the following way. Let it be supposed that the ocean-spheroid assumes a form agreeing with the equilibrium-spheroid at the moment, and that the pole of this spheroid follows the position of the pole of the equilibrium-spheroid at a certain mean interval, say 90°. Let it be supposed that at a certain time a tide is sent off from this ocean-spheroid along a channel in which it is no longer affected by the moon or sun, and thus reaches Liverpool, producing the tide there. The following assumptions will then represent the facts.

When the horizontal parallax is 54′, the tide is sent off along the channel in longitude $48\frac{1}{2}°$ east of Liverpool, at $45^h\ 6^m$ before the time of Liverpool high water, and the pole of the ocean-spheroid follows the position of equilibrium at a distance 90° 24′.

When the horizontal parallax is 57′, the tide is sent off along the Liverpool channel in longitude $94\frac{1}{2}°$ east, at $48^h\ 36^m$ before the Liverpool high water, and the actual spheroid is 90° behind the position of equilibrium.

When the horizontal parallax is 61′, the tide is sent off along the channel in longitude $159\frac{1}{2}°$ east, at $53^h\ 0^m$ before the occurrence of high water, and the actual spheroid is 89° 16′ behind the position of equilibrium.

This hypothesis thus modified represents the circumstances of the Liverpool tide as affected by lunar parallax. The effect of lunar declination might be represented in a similar manner.

It is not to be imagined that this hypothetical representation is near to the true state of the case. The changes in the lagging, in the length of the channel of transmission, and in the velocity of transmission, are not such as the forces can be supposed likely to produce. Nor is it likely that the original tide is exactly what it would be if the condition of equilibrium were fully attained. The tide-spheroid not only lags behind the position of equilibrium, but deviates from the form of equilibrium; and other differences, besides the retroposition in longitude and in time, are introduced by the waters being in motion instead of at rest. This is seen in our results; for the tidal force of the moon, which, in the equilibrium-spheroid, varies as the cube of the parallax, appears in the observations to vary more nearly as the square of the parallax: and though this difference may be referred to the inaccuracy of the observations, it may, I think with more probability, be considered as resulting from the condition of the waters being a condition of motion, not of equilibrium. The temporary variations of the force do not affect the form of the waters in the same proportion as the mean force, which is constantly dragging the waters after it, round the earth.

11. In what has been hitherto stated with regard to the hypothetical representation of the tides, we have had a reference solely to Liverpool. It cannot, however, be doubted that the general laws of the tides at other places would resemble those of that port, and therefore might be represented in a similar manner. It has already been shown in my former memoir, though less satisfactorily and precisely than in this, that the tides of London follow the same rules as those now described.

The numbers, however, which enter into these formulæ will not necessarily be the same at two places; and since the empirical formulæ have not been determined for any places except those of London and Liverpool, we have not the means of discovering the relation of the constants at various places. The following comparison of the data of observation at London and at Liverpool is instructive as far as it goes.

The greatest difference arising from the mean semimenstrual inequality is the same at the two ports, being 88^m at both. This coincidence is striking; yet I am disposed to believe it accidental, although, according to theory, this quantity ought to be the same at all places, since it depends only upon the mean ratio of the solar and lunar tidal forces; for the semimenstrual inequalities at different places differ so much by other observations, (varying from 79^m at Brest to 96^m at Plymouth,) that I do not conceive the difference can arise from the incompleteness of the observations.

The effects of the parallax and declination at London were given by approximate formulæ, less exact than those which we have now obtained for Liverpool; but, comparing the London formulæ with corresponding approximations at Liverpool, we have the following results.

If Λ' represent the value of λ' for the mean parallax $57'$ and the declination $0°$, we have for the parallax p, and declination δ,

at London $\lambda' = \Lambda' - 3^m (p - 57) - 132^m \sin^2 \delta$,

at Liverpool $\lambda' = \Lambda' - 2\frac{1}{2}^m (p - 57) - 84^m \sin^2 \delta$, by Art. 15 and 21.

Also the maximum semimenstrual inequality,

at London $= 40^m + 3^m (p - 57) + 84^m \sin^2 \delta$,

at Liverpool $= 41^m + 2^m (p - 57) + 30^m \sin^2 \delta$, by Art. 17 and 23.

Also if H' be the value of h' for the mean parallax and the declination $0°$, we have,

at London $h' = H' + 0^{ft}{\cdot}17 (p - 57) - 3^{ft} \sin^2 \delta$,

at Liverpool $h' = H' + 1^{ft}{\cdot}47 (p - 57) - 6^{ft} \sin^2 \delta$, by Art. 19 and 24.

And at London $h = 1^{ft}{\cdot}7$.

at Liverpool $h = 2^{ft}{\cdot}8$.

12. The resemblances of the formulæ at the two places are remarkable, but the differences are still more so. The differences in the heights of the tide at different places are indeed what we know to prevail universally, and to depend upon local circumstances in an intelligible manner: but the differences in time are more difficult to explain, since both the tides come from the same origin. The difference in the effect of parallax may indeed be due to the inaccuracy of the data, but it is scarcely possible that this can be true of the difference in the effect of declination, which appears to be in the ratio of 132 to 84 for the non-periodical, and 84 to 30 for the periodical, part. Similar discussions of observations at other places will best throw light on this difficulty.

I now proceed to state the method by which the above empirical formulæ have been obtained.

The Semimenstrual Inequality.

13. In order to obtain the semimenstrual inequality of the *times* of high water, I take Mr. LUBBOCK's Table VII., and from each column of intervals (of tide and moon's transit) I subtract the mean of that column; and I thus obtain Table VII. (a), which exhibits the semimenstrual inequalities for each minute of parallax. I then take the means of the horizontal lines in this, interpolating in H. P. 60′ and omitting H. P. 61′. The resulting intervals are those of the mean tide.

TABLE VII. (a.)

Mean of each column subtracted from the column "*Interval*" of times.

H. P.......	54′.	55′.	56′.	57′.	58′.	59′.	60′.	61′.	Mean.
Mean Interval	h m 11 12·7	h m 11 11·5	h m 11 8·3	h m 11 6·5	h m 11 3·7	h m 11 0·3	h m 10 58·5	h m 10 54·5	h m 11 6
☽'s Transit.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.
h m									
0 30	+13·7	+11·6	+ 8·0	+11·6	+12·2	+12·9	+13·3	+11·8	+12·2
1 30	− 5·2	− 6·8	− 6·5	− 3·9	− 5·3	− 2·8	− 2·5	− 2·8	− 4·6
2 30	−23·2	−22·0	−22·0	−20·8	−18·6	−17·5	−16·6	−17·1	−20·0
3 30	−41·4	−36·7	−34·7	−33·0	−32·1	−29·3	−30·4	..	−33·9
4 30	−49·0	−47·8	−44·0	−41·8	−40·4	−38·4	−38·7	..	−42·8
5 30	−47·7	−45·7	−43·6	−43·2	−39·4	−38·5	−37·5	..	−43·2
6 30	−27·2	−26·4	−24·5	−25·7	−25·6	−24·8	−21·5	..	−25·0
7 30	+14·3	+13·4	+11·9	+ 9·2	+ 6·5	+ 1·6	+11	..	+ 9·6
8 30	+44·8	+41·8	+40·1	+37·4	+34·1	+31·1	+20·2	..	+36·6
9 30	+50·9	+49·3	+47·9	+45·0	+44·1	+41·6	+39·8	+39	+45·6
10 30	+42·5	+41·8	+41·3	+40·1	+38·6	+38·0	+36·1	+35·8	+39·8
11 30	+28·5	+28·1	+26·6	+25·7	+23·6	+25·7	+24·4	+25·4	+26·1
Max. Diff.	99·9	95·1	91·9	88·2	84·5	80·1	78·5		88·8

On comparing the mean numbers in the last column with the theoretical formula

$$\tan 2\,(\theta' - \lambda') = -\frac{c \sin 2\,(\phi - \alpha)}{1 + c \cos 2\,(\phi - \alpha)},$$

it appears that they may be very accurately represented by making $\lambda' = 11^{\mathrm{h}}\,6^{\mathrm{m}}$, $\alpha = 1^{\mathrm{h}}\,15^{\mathrm{m}}$, $c = \sin 1^{\mathrm{h}}\,29^{\mathrm{m}}$. The agreement of this formula with observation is as follows:

Moon's Transit.	Formula.	Obs.	Diff.
h m	m s	m s	m s
0 30	+12 16	+12 12	− 0 4
1 30	− 4 7	− 4 36	− 0 29
2 30	−20 6	−20 0	+ 0 6
3 30	−34 0	−33 54	+ 0 6
4 30	−43 6	−42 48	+ 0 18
5 30	−42 40	−43 12	− 0 32
6 30	−25 8	−25 0	+ 0 8
7 30	+ 9 2	+ 9 6	+ 0 4
8 30	+36 28	+36 36	+ 0 8
9 30	+44 30	45 36	+ 1 6
10 30	+39 40	39 48	+ 0 8
11 30	+27 36	26 6	− 1 30

This accordance is complete, the difference amounting in only two cases to 1^{m}.

14. The semimenstrual inequality of the *heights* for each minute of horizontal parallax, and the mean semimenstrual inequality of the heights, are in like manner obtained by subtracting from each column of heights in Mr. Lubbock's Table VII. the mean of that column, and taking the means of the horizontal lines as is done in Table VII. (a.)

Table VII. (b.)

Mean of each column subtracted from column "*Height* of Tide."

H. P.......	54′.	55′.	56′.	57′.	58′.	59′.	60′.	61′.	Mean.
Mean height	14·20	14·41	14·84	15·22	15·63	16·02	16·43	16·66	
☽'s Transit. h m	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.
30	+2·38	+2·47	+2·45	+2·31	+2·33	+2·35	+2·19	+2·51	+2·35
1 30	+2·26	+2·23	+2·25	+2·41	+2·49	+2·43	+2·66	+2·73	+2·39
2 30	+1·62	+1·61	+1·83	+1·82	+1·91	+2·02	+2·32	+2·21	+1·88
3 30	+0·49	+0·70	+0·74	+0·95	+0·95	+1·17	+1·27		+ ·90
4 30	−0·61	−0·55	−0·64	−0·33	−0·27	−0·15	−0·10		− ·38
5 30	−1·92	−1·92	−1·93	−1·83	−1·62	−1·53	−1·57		−1·76
6 30	−3·13	−3·05	−3·14	−2·69	−2·79	−2·73	−2·85		−2·91
7 30	−2·81	−2·96	−2·74	−2·89	−2·97	−3·06	−3·14		−2·94
8 30	−1·52	−1·62	−1·69	−1·86	−1·86	−2·09	−2·33		−1·85
9 30	−0·06	−0·08	−0·18	−0·43	−0·54	−0·60	−0·75	−0·60	− ·38
10 30	+1·18	+1·16	+1·13	+0·82	+0·80	+0·60	+1·61	−0·65	+1·04
11 30	+2·10	+1·89	+1·95	+1·78	+1·63	+1·64	+1·65	+1·78	+1·81
Max. diff.	5·51	5·52	5·59	5·30	5·46	5·49	5·80	......	5·33
Mean ..	5·44								
Half ...	2·74								

The theoretical height of the high water above the mean surface of the ocean is $\sqrt{\{h'^2 + h^2 + 2\,h\,h' \cos 2\,(\varphi - \alpha)\}}$; and therefore if k be the mean of all the high-water heights, we shall have for the semimenstrual inequality of height the expression

$$\sqrt{\{h'^2 + h^2 + 2\,h\,h' \cos 2\,(\varphi - \alpha)\}} - k.$$

This will agree very nearly with the result of observation if we make

$$h = 2{\cdot}74,\ h' = 6{\cdot}872,\ k = 7{\cdot}19,\ \alpha = 1^{h}.$$

The accordance is as follows:

Moon's Transit.	Formula.	Obs.
h m	f.	
0 30	2·35	2·35
1 30	2·35	2·39
2 30	1·83	1·88
3 30	0·84	0·90
4 30	−0·48	−0·38
5 30	−1·89	−1·76
6 30	−2·90	−2·91
7 30	−2·90	−2·94
8 30	−1·89	−1·85
9 30	−0·48	−0·38
10 30	0·84	1·04
11 30	1·83	1·81

The greatest deviation is about an inch, the mean a small fraction of an inch.

Effect of the Moon's Parallax.

15. In Mr. Lubbock's Table VII., which contains the effect of lunar parallax, and has a column for each minute of parallax, we have, in Art. 13, taken the mean of each column, and subtracted it from every number in the column. In this way, it is evident that the *mean* contains the *non-periodical* part of the effect, and the *remainder* contains the part which goes through its *period* in a semi-lunation.

The *non-periodical* part of the *interval* stands in the uppermost line of Table VII. (a) Art. 13.; and its variations are manifestly nearly or exactly proportional to the variations of the parallax. If we take 57′ as the mean parallax, we may express these means very nearly by the formula

$$11^h\ 6^m{\cdot}5 - 2{\cdot}5\ (p - 57'),$$

p being the H. P. The agreement of the formula with observation is as follows, and is a near approximation.

H. P.	54′.	55′.	56′.	57′.	58′.	59′.	60′.	61′.
	h m	h m	h m	h m	h m	h m	h m	h m
Obs.	11 12·7	11 11·5	11 8·3	11 6·5	11 3·7	11 0·3	10 5·85	10 54·5
Formula....	11 13	11 11·5	11 9	11 6·5	11 4	11 1·5	10 5·9	10 56·5

The column for H. P. 60′ is completed by interpolation, and the column for H. P. 61′ is omitted. The latter is defective in half the hours of moon's transit, which arises from the effect of the Moon's Variation on the parallax. The parallax has a term depending on the sine of twice the distance of the moon from the sun, and cannot be so great as 61′ except near syzygy. The "observed" mean for 61′ is that which makes the numbers in that column follow nearly the same law as the rest.

The *periodical* part of the effect of lunar parallax is shown in the lower part of Table VII. (a). It appears there that the *intervals* for all the values of H. P. follow nearly the same law as the mean interval already considered, but with a difference in the maximum value of the inequality. If we add together the greatest positive and negative numbers in each column, we obtain the double of the maximum inequality nearly, but not exactly, since the maximum does not correspond exactly to times of moon's transit contained in the Table. Making a slight addition on this account, we have,

H. P............	54′.	55′.	56′.	57′.	58′.	59′.	60′.	61′.
	m	m	m	m	m	m	m	
Sum	99·9	95·1	91·9	88·2	84·5	80·1	78·5	
Double Max.	101	96·1	92·8	89	85·2	80·7	79	
Formula....	101	97	93	89	85	81	77	

Now in the expression

$$\tan 2\,(\theta' - \lambda') = -\frac{c \sin 2\,(\phi - \alpha)}{1 + c \cos 2\,(\phi - \alpha)}$$

the maximum value occurs when $\cos 2\,(\phi - \alpha) = -c$, and is equal to $\frac{c}{\sqrt{1 - c^2}}$. If we make $c = \sin\gamma$, we have for the maximum $\tan 2(\theta' - \lambda') = \tan\gamma$; and therefore maximum $2(\theta' - \lambda') = \gamma$. Hence we have γ for each H. P. by reducing the above double maxima to arcs, and thence we have c, by finding the sines of these arcs.

16. By the equilibrium-theory, c should be inversely as the cube of the H. P.; therefore, if C be the value of c for H. P. 57′, we have $\frac{c}{C} = \frac{57^3}{p^3}$; $\log c + 3\log p = \log C + 3\log 57$; and therefore this quantity, $\log c + 3\log p$, should be constant. It is found that we get a quantity much more nearly constant by taking $\log c + 2{\cdot}2 \log p$. The following is the result:

H. P.	γ.	$\log c$ ($c = \sin\gamma$).	$\log p$.	$\frac{22}{10}\log p$.	$\log c\,p^{2\cdot2}$.
54′.	25° 15′	9·62999	1·73239	346478 3·46478	3·44125
55.	24 15	9·61354	1·74036	348072 3·48072	3·44233
56.	23 12	9·59543	1·74819	349638 3·49638	3·44145
57.	22 15	9·57824	1·75587	351174 3·51174	3·44115
58.	21 18	9·56021	1·76343	352686 3·52686	3·43976
59.	20 10	9·53751	1·77085	354170 3·54170	3·43338
60.	19 45	9·52881	1·77815	355630 3·55630	3·44074
61.					

Hence, $c = C\left(\frac{57}{p}\right)^{2\cdot2}$ nearly.

17. We may, however, express the result more conveniently for some purposes by expanding this expression; for the variation of the maximum will be very nearly as the variation of the parallax; and the double maximum may be nearly expressed by the following formula:

$$89^{\mathrm{m}} - 4^{\mathrm{m}}(p - 57).$$

The accordance is shown in the lowest line of the second Table in Art. 15.

18. By comparing, in Table VII. (a), the inequalities for moon's transit $0^{\mathrm{h}}\,30^{\mathrm{m}}$, $1^{\mathrm{h}}\,30^{\mathrm{m}}$, and for $6^{\mathrm{h}}\,30^{\mathrm{m}}$ and $7^{\mathrm{h}}\,30^{\mathrm{m}}$, it is clear that they are equal to 0 at a later hour for the larger than for the smaller parallaxes, which also appears by the maxima. Hence α is larger for large parallaxes than for small ones. The exactness of the observations hardly allows us to determine its variation exactly. It appears, however, that it may be sufficiently well represented by $\alpha = 1^{\mathrm{h}}\,15^{\mathrm{m}} + 2^{\mathrm{m}}{\cdot}5\,(p - 57)$.

19. The effect of the lunar parallax on the *heights* will be found from Table VII. (b.) in the same way as the effect on the times, by taking the *mean* of each column as the *non-periodical*, and the *remainder* as the *periodical*, part of the inequality. The origin of the measurements is arbitrary, the low water not being given. The *non-periodical* part is represented with great accuracy (except for the extreme parallaxes) by the formula $15{\cdot}22 + {\cdot}4\,(p - 57)$. The accordance is as follows:

H. P. ...	54′.	55′.	56′.	57′.	58′.	59′.	60′.	61′.
Obs. ..	14·20	14·41	14·84	15·22	15·63	16·02	16·43	16·66
Formula	14·02	14·42	14·82	15·22	15·62	16·02	16·42	16·82

We cannot compare this effect of parallax on the heights with the whole height of the tide, or with theory, from not having any observations of low water for this series of tides.

The periodical part of the heights, as appears by the remainder of Table VII. (b.), whatever be the parallax, follows nearly the law of the mean, which has already been explained; and the magnitude of the maximum differences does not appear to be steadily different for different H. P. In fact, theory would lead us to expect it to be the same in all these cases, because the amount of this inequality is 2 h, double the mean solar tide.

20. But it appears from the Table VII. (b.) that the time of moon's transit, when the periodical inequality vanishes, is later for the larger parallaxes, and the maxima indicate the same change: the amount of the change is about 4^{m} $(p - 57)$, at the mean between the greatest and least values of the height.

When $\varphi = \frac{\pi}{4} + \alpha$, the formula for the inequality becomes $h' - h$, which is the mean between the greatest and least values. In this case $\alpha = 1^{h}$. Hence the value of α is $60^{m} + 4^{m}$ $(p - 57)$.

Effect of the Moon's Declination.

21. The effect of the changes of lunar declination upon the tide will be found in nearly the same way as the effect of changes in the parallax. Mr. Lubbock's Table XII. gives the *intervals* for each 3° of declination. By finding the mean of each column, and subtracting it from the column, we obtain the *non-periodical* and the *periodical* part respectively of the inequality as is done in Table XII. (a.)

Table XII. (a.)

[*Intervals* of times.] Mean of each column subtracted from the column.

Decl.	0°.	3°.	6°.	9°.	12°.	15°.	18°.	21°.	24°.	27°.
Mean Interval	h m 11 12·1	h m 11 11·0	h m 11 11·3	h m 11 10·4	h m 11 8·4	h m 11 6·5	h m 11 3·6	h m 11 1·2	h m 10 59·0	h m 10 55·6
☽'s Transit. h m	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.
0 30	+10·0	+ 8·4	+ 8·8	+ 9·1	+ 9·5	+10·8	+12·1	+12·3	+13·2	+13·6
1 30	− 7·1	− 6·3	− 5·6	− 6·1	− 5·5	− 5·3	− 4·6	− 3·2	− 3·8	− 3·3
2 30	−20·7	−20·4	−20·5	−21·1	−20·4	−20·5	−20·0	−20·8	−20·8	−20·6
3 30	−33·0	−31·9	−32·7	−31·5	−33·0	−33·7	−30·4	−33·0	−34·4	−33·7
4 30	−40·5	−41·2	−40·9	−40·3	−41·1	−42·2	−42·0	−43·3	−44·6	−45·7
5 30	−37·6	−36·0	−37·2	−38·3	−40·2	−40·4	−43·4	−43·5	−45·7	−46·6
6 30	−17·4	−17·2	−19·6	−20·7	−22·2	−23·1	−25·3	−26·9	−28·4	−31·7
7 30	+11·3	+14·0	+11·5	+12·0	+10·0	+11·4	+ 8·6	+ 3·4	+ 9·3	+ 4·3
8 30	+35·0	+33·0	+37·1	+34·3	+37·2	+36·4	+36·5	+39·7	+37·3	+40·9
9 30	+41·7	+41·2	+44·7	+43·1	+42·2	+43·6	+44·1	+46·0	+46·6	+49·9
10 30	+36·0	+31·8	+34·7	+36·0	+38·5	+37·9	+38·5	+40·9	+42·1	+43·6
11 30	+22·3	+24·6	+21·9	+23·9	+25·6	+25·6	+26·6	+28·2	+28·9	+30·0
Greatest Diff.	82·2	82·4	83·6	83·4	83·3	85·8	87·5	89·5	92·3	96·5
Excess above 82	0·2	0·4	1·6	1·4	1·3	3·8	5·5	7·5	10·3	14·5

The first line of that Table contains the *non-periodical* part. In order to find its law, subtract each mean from $11^h\ 12^m$ corresponding to decl. 0. We obtain a series of numbers which increase faster than the declination; and it is found that they may be nearly represented by the expression $84 \sin^2 \delta$, δ being the declination. The agreement is as follows:

Declination......	0°.	3°.	6°.	9°.	12°.	15°.	18°.	21°.	24°.	27°.
Obs. Diff.....	0	1·0	0·7	1·7	3·6	5·5	8·0	10·8	13·0	16·4
Formula	0	0·2	0·8	2·0	3·6	5·6	8·0	10·8	13·9	17·3

Hence the *non-periodical* part is $11^h\ 12^m - 84^m \sin^2 \delta$.

The *remainder* of the Table XII. (a.) exhibits the periodical part of the inequality; and it will be seen that each column follows nearly the law of the mean semimenstrual inequality as already obtained. In order to obtain the law of the coefficients, I take, as before, the sum of the two maximum values. This sum converted into arc gives γ, and $c = \sin \gamma$.

Decl.	γ.	$\operatorname{cosec} \gamma = \frac{1}{c}$.	Excess of Decl. 0°.	Log. Excess.	Log. $\sin^2 \delta$.	Difference.
° 0	° ′ 20 33	2·8488028				
3	20 36	2·8421877				
6	20 54	2·8031777				
9	20 51	2·8091995	·0396033	$\bar{2}$·59769	18·38866	·20903
12	20 50	2·8117471	·0370557	$\bar{2}$·56878	18·63576	$\bar{1}$·93302
15	21 27	2·7345630	·1142398	$\bar{1}$·05778	18·82600	·23178
18	21 52	2·6849391	·1638637	$\bar{1}$·21447	18·97996	·23451
21	22 23	2·6260406	·2227622	$\bar{1}$·34783	19·00866	·33917
24	23 5	2·5505680	·2982348	$\bar{1}$·47455	19·21862	·25593
27	24 8	2·4458163	·4029865	$\bar{1}$·60528	19·31410	·29118

For the smaller declinations, the differences are too small to be depended on. The numbers corresponding to the resulting logarithms from 15° to 27° are from 1·7 to 2·1. If we take the mean 1·85 as the number, we have for $\frac{1}{c}$ the value $2{\cdot}85 - 1{\cdot}85 \sin^2 \delta$, which is sufficiently near.

22. By the theory of equilibrium $\frac{1}{c} = \frac{h'}{h}$. And by the same theory, if H' be the height of the lunar tide at the equator when the declination is 0, we shall have in latitude l, when the declination is δ, two tides, of which the heights are $H' \cos^2 (l + \delta)$ and $H' \cos^2 (l - \delta)$. Now as we have not distinguished these two tides, our result will be the mean of them. Therefore,

$$\begin{aligned} h' &= \tfrac{1}{2} H' \{\cos^2 (l + \delta) + \cos^2 (l - \delta)\} \\ &= H' \{\cos^2 l \cos^2 \delta + \sin^2 l \sin^2 \delta\} \\ &= H' \{\cos^2 l - (\cos^2 l - \sin^2 l) \sin^2 \delta\} \\ &= H' \cos^2 l \{1 - (1 - \tan^2 l) \sin^2 \delta\}. \end{aligned}$$

But we have found that h' varies as

$$2{\cdot}85 - 1{\cdot}85 \sin^2 \delta.$$

Hence $1 - \tan^2 l = \frac{1{\cdot}85}{2{\cdot}85} = \frac{37}{57}$; $\tan^2 l = \frac{20}{57}$

$$l = 30^\circ\ 38'.$$

Hence it appears that the amount of the effect of declination on the time H. W. at Liverpool corresponds to the effect on an equilibrium-spheroid at a place in latitude 30°.

23. It will be convenient to express the double maximum of the semimenstrual inequality by a formula. The formula $60^m \sin^2 \delta$ gives the following accordance with the excess above 82^m:

Decl.	0°.	3°.	6°.	9°.	12°.	15°.	18°.	21°.	24°.	27°.
Obs. ...	0·2	0·4	1·6	1·4	1·3	3·8	5·5	7·5	10·3	14·5
Formula	0	0·2	0·3	1·4	2·6	4·0	6·0	7·7	9·9	12·4

Hence $82^m + 60^m \sin^2 \delta$ nearly expresses the double maximum.

It appears also that the times when the inequality is 0 and max. are later as the declination is greater, the difference being about 15^m from decl. 0° to decl. 27°, and apparently increasing as the square of the decl. or of sin decl. Now $\sin^2 27 = \frac{1}{5}$ nearly: hence the correction of α is nearly $75^m \sin^2 \delta$, and we have $\alpha = 65^m + 75^m \sin^2 \delta$.

24. The effect of lunar declination on the *heights* will be found in the same manner from Mr. LUBBOCK's Table XVI. The mean of each column is obtained in the first line of the following Table.

TABLE XVI. (b.)

[Heights.] Mean of each column subtracted from the column.

Decl.	0°.	3°.	6°.	9°.	12°.	15°.	18°.	21°.	24°.	27°.
Mean Height ...	15·74	15·78	15·77	15·72	15·54	15·41	15·23	14·92	14·74	14·39
☽ 's Transit.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.	Remainder.
h m										
0 30	+2·59	+2·64	+2·30	+2·52	+2·65	+2·58	+2·27	+2·53	+2·18	+2·38
1 30	+2·53	+2·52	+2·73	+2·57	+2·61	+2·51	+2·46	+2·32	+2·34	+2·40
2 30	+1·75	+1·97	+1·95	+1·82	+1·99	+1·81	+2·03	+1·86	+1·99	+2·10
3 30	+1·01	+0·69	−0·77	+0·88	+0·80	+0·84	+1·10	+1·03	+1·21	+1·21
4 30	−0·70	−0·48	−0·53	−0·74	−0·54	−0·52	−0·70	−0·27	−0·22	−0·22
5 30	−1·62	−2·10	−1·85	−1·96	−1·83	−1·81	−2·01	−1·86	−1·75	−1·61
6 30	−2·66	−2·74	−3·00	−2·86	−3·08	−2·96	−2·88	−3·12	−3·05	−3·24
7 30	−2·83	−2·66	−2·70	−3·10	−2·95	−3·02	−2·91	−3·01	−3·10	−3·20
8 30	−1·61	−2·28	−2·11	−1·72	−1·89	−1·99	−1·90	−1·95	−1·94	−1·88
9 30	−0·60	−0·37	−0·42	−0·32	−0·53	−0·41	−0·30	−0·40	−0·21	−0·39
10 30	+1·12	+0·92	+1·03	+0·84	+0·95	+1·12	+0·99	+1·01	+0·80	+0·85
11 30	+1·09	+1·93	+1·84	+2·14	+1·87	+1·81	+1·90	+1·87	+1·78	+1·60
Greatest Diff.	5·42	5·38	5·73	5·67	5·73	5·60	5·37	5·65	5·44	5·64
Mean	5·63									

If we subtract these mean heights from 15·8, the remainders are very nearly as $\sin^2 \delta$. The formula $6 \sin^2 \delta$ gives the following accordance:

Decl.	0°.	3°.	6°.	9°.	12°.	15°.	18°.	21°.	24°.	27°.
Obs.	·06	·02	·03	·08	·26	·39	·57	·88	1·06	1·41
Formula . .	·00	·02	·08	·14	·26	·40	·60	·77	·99	1·24

Hence $15·8 - 6 \sin^2 \delta$ nearly is the non-periodical part of the Table.

The *periodical* part, as appears by the remainder of Table XVI. (b.), follows nearly the law of the mean height already explained. The sum of the maximum inequalities is not definitely different for the different declinations, which agrees with the theory, according to which it is constant and equal to $2h$.

Also by comparing the columns for decl. 0° and 27°, it appears that the interval between the times when the inequality is 0, is less for the greater decl., which also agrees with the theory, for in that case the fraction $\frac{h}{h'}$ is greater, and the defect of symmetry in the curve increases with this fraction.

There is no clear evidence of a variation of α in this Table.

Trinity College, Cambridge,
November 12, 1835.